On Λ-positioning of an arc between two parallel support lines

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Abstract

We show that a rectifiable plane arc g has two parallel support lines and a triple of consecutive points $g(r),\ g(s),\ g(t),\ \ r < s < t$, so that $g(s)$ lies on one line, while $g(r)$ and $g(t)$ lie on the other. If the arc is simple, such a pair of lines is unique. [1]

Introduction. In the articles [1, 3] we had to use the result in [2]: *If a convex set covers any simple polygonal unit arc, it covers any unit arc.* In [1, 3] the requirement on an arc to be simple and polygonal was used only in Theorem 5.1 of [1] establishing that *any simple polygonal arc assumes a so-called* Λ*-configuration* (Figure 1). Two proofs of Theorem 5.1 exist for simple arcs: one by Y. M. (Geometry Seminar, UIUC, 2009) and the other by R. Alexander, J. E. Wetzel, W. Wichiramala in their recently submitted paper "The Λ-property of a simple arc". In this note we prove Theorem 5.1 of [1] omitting both requirements: simple and polygonal.

Given a parametrization $g(s),\ s \in [0,1]$, for points $p = g(s_1),\ q = g(s_2)$ with $s_1 < s_2$, we say that p *precedes* q and write it as $p \prec q$. Points $p_1,\ p_2,\ p_3$ form a *triple of consecutive points* if either

$$p_1 \prec p_2 \prec p_3 \quad \text{or} \quad p_3 \prec p_2 \prec p_1. \tag{1}$$

We are seeking a pair of parallel support lines with and a triple of consecutive points $p_1,\ p_2,\ p_3$ such that p_1, p_3 lie on one line and p_2 lies on the other.

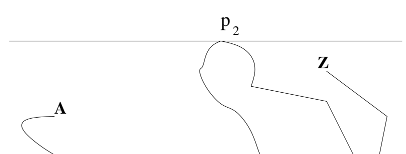
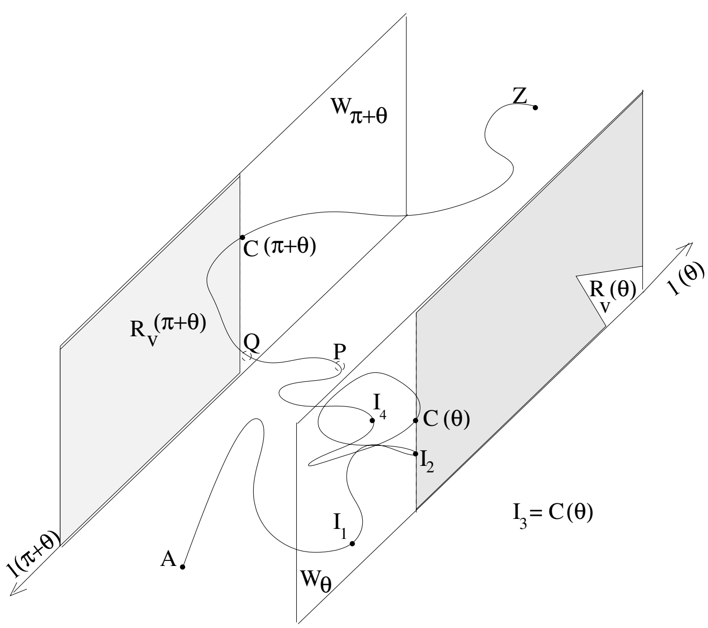


Figure 1. **Λ-configuration**.

[1] *AMS classification:* 52C15 *Keywords:* rectifiable arc, support line

This is clearly true when g is closed or is a straight segment.

Lines $l(\theta)$, walls $W(\theta)$ and touch points in Figure 2. Let $g(s) : [0,1] \to R^2$ be an open rectifiable plane arc in the horizontal (x,y)-plane, whose thickness h is positive. We denote by $l(\theta)$ a counter-clockwise oriented support line for g in the direction of polar angle θ and assume that g lies above $l(0)$. Let $G(s) = (g(s), s) : [0,1] \to R^2 \times [0,1]$ be a simple lift of g to R^3 with $G(0) = A$, $G(1) = Z$. Given a direction θ, let us denote by $W(\theta)$ the support plane (wall) to G through $l(\theta)$ orthogonal to the (x,y)-plane. The touch points $G \cap W(\theta)$ are denoted by I_k. Note that a set of touch points of a wall $W(\theta)$ is its compact subset. The wall $W(\theta)$ has its lowest most left $I_1(\theta)$ and highest most right $C(\theta)$ touch points with respect to the counter-clockwise orientation of the support line $l(\theta)$. Let $v(\theta)$ be the vertical unit segment through $C(\theta)$.



Touch points $I_k(\theta) \in W(\theta)$, $I_m(\theta+\pi) = C(\theta+\pi) \in W(\theta+\pi)$. For all k,m, $I_k(\theta) \prec I_m(\theta+\pi)$.

Figure 2

Rotation of a support wall as a map. Note that for each $\theta \in [0, 2\pi)$, there is one support line $l(\theta)$ and thus one support wall $W(\theta)$. Referring to a

rotation of a support line around g as θ changes from 0 to 2π, we think of two maps from $[0, 2\pi)$: one is the map to the set of all lines in the (x, y)-plane and the other is the map to all planes in the (x, y, z)-space. The images of such maps are the set of oriented support lines $l(\theta)$ of g and the set of support walls $W(\theta)$ through $l(\theta)$. Geometrically the first map is represented by a line moving in a plane so that it coincides with $l(\theta)$ for each θ, while the second map is represented by a plane moving in the space so that it coincides with $W(\theta)$ for each θ.

Local stability of a non-Λ-configuration. A subarc of g between points X and Y is denoted by $\widetilde{XY}$.

Lemma 1. *The following set* $\Theta = \{\theta \geq 0 : \ C(\theta) \prec C(\theta+\pi) \ \}$ *is a half-open interval. That is if* $\theta \in \Theta$, *then there exist* δ *so that* $\theta + \varepsilon \in \Theta$ *for any* $\varepsilon < \delta$.

Proof. To keep our proof transparent, we assume that $v(\theta)$ has only finitely many touch points and we will use a particular configuration of Figure 2. Let points $P, Q \in \widetilde{I_4D}$ be so that

(2) $I_4(\theta) \prec P \prec Q \prec C(\theta+\pi)$ *and* $length(\widetilde{I_4P}) = length(\widetilde{QD}) = \dfrac{1}{3}h.$

Denote by $R_v(\theta) \subset W(\theta)$ a half-plane of points to the right of $v(\theta)$. Let

$$\sigma(\theta) = \min\left\{\mathrm{dist}[\widetilde{\mathrm{PZ}}, \mathrm{R_v}(\theta)],\ \mathrm{dist}[\widetilde{\mathrm{AQ}}, \mathrm{R_v}(\pi+\theta)]\right\} .$$

Then $\sigma > 0$ and $\mathrm{dist}(\widetilde{\mathrm{PZ}}, \mathrm{v}(\theta))$ and $\mathrm{dist}(\widetilde{\mathrm{AQ}}, \mathrm{v}(\theta+\pi))$ are $\geq \sigma$. We take any

$$\varepsilon \ < \ \delta = \frac{\sigma(\theta)}{88\ diameter(G)}.$$

The obstacles to the counter-clockwise rotation of the walls by ε could be only subarcs $\widetilde{QZ}$ and $\widetilde{AP}$. Therefore, $C(\theta+\varepsilon) \in \widetilde{AP}$ while $D(\theta+\varepsilon) \in \widetilde{QZ}$ and hence by (2), $C(\theta+\varepsilon) \prec D(\theta+\varepsilon)$. ♦

Theorem 1. *Let g be an open rectifiable arc with a thickness $h > 0$. Then there exist support lines $l(\theta)$ and $l(\pi+\theta)$ containing a triple of consecutive points $p_1,\ p_2,\ p_3$ of g with the lone middle point p_2.*

Proof. We may assume that $A \prec C(0), C(\pi) \prec Z$. (Figure 2). If a triple of the theorem exists in the strip between $l(0)$ and $l(\pi)$ then $C(\pi) \prec C(0)$ (Figure 1). Otherwise,

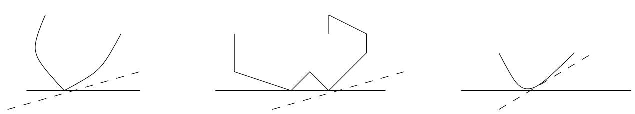
$$C(0) \prec C(\pi). \tag{3}$$

By Lemma 1 this property is locally stable and so if Θ is the set given by this lemma and $\boldsymbol{\theta_T = lub(\Theta)}$, then $\theta_T \notin \Theta$. That is $C(\theta_T + \pi) \prec C(\theta_T)$. However, limits of most right touch points in Θ preserve this property of Θ :

$$\lim_{\theta \uparrow \theta_T} C(\theta_T) \prec \lim_{\theta \uparrow \theta_T} C(\theta_T + \pi), \tag{4}$$

because they are separated in distance by h. Indeed,

Small rotations around G by a small angle ψ. Configurations of local behavior near a touch point are given in Figure 3:



***I*(θ+ψ)=*I*(θ) in rotations without an obstacle, dist(I(θ+ψ),I(θ)) is small in rotations around smooth convex arcs.**

Figure 3

Thus one or both $C(\theta_T)$ or $C(\theta_T + \pi)$ are not limits in (4). Suppose that $C(\theta_T)$ is not equal to $\lim\limits_{\theta \uparrow \theta_T} C(\theta_T)$ Then the triples $p_1 \prec p_2 \prec p_3$ satisfying the theorem are either

$$\lim_{\theta \uparrow \theta_T} C(\theta_T) \prec \lim_{\theta \uparrow \theta_T} C(\theta_T + \pi) \prec C(\theta_T)$$

or

$$C(\theta_T + \pi) \prec C(\theta_T) \prec \lim_{\theta \uparrow \theta_T} C(\theta_T + \pi).$$

An assumption that there were no $\boldsymbol{\theta_T \leq \pi}$ leads to a contradiction. Suppose that $\theta_T > \pi$, that is $C(\pi) \prec C(\pi + \pi)$. On the other hand, after a rotation by π, we arrive to the initial position of g between the same parallel support lines. In this configuration, the highest most right touch point on

the wall $W(\pi)$ is a successor to such point on the wall $W(\pi+\pi) = W(0)$ and (3) is true: $C(\pi + \pi) = C(0) \prec C(\pi)$. That is a contradiction. ■

Corollary (W. Wichirimala). *If g is simple, then the pair of support lines with g positioned as in Figure 4 is unique.*

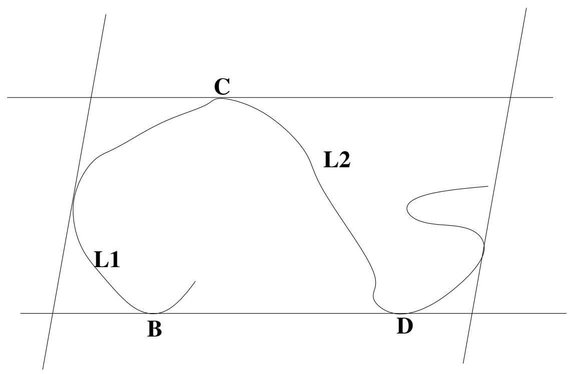


Figure 4

Proof. In Figure 4, the point C divides the curve into two curves $L1$ and $L2$ where each point on $L1$ is parametrically precedes each point on $L2$. Consider a different pair of two parallel support lines. All touch points on one, say, the left line belong to $L1$, therefore none of them can be parametrically between two touch points on the right line belonging to $L2$. ♦

References

[1] P. Coulton and Y. Movshovich, Besicovitch triangles cover unit arcs, *Geom. Dedicata* **123** (2006), 79-88.

[2] J. M. Maki, J. E. Wetzel, and W. Wichirimala, Drapeability, *Discrete Comput. Geom.,* **34** (2005), 637-657.

[3] Y. Movshovich, Besicovitch triangles extended, *Geometriae Dedicata* **159(1)** (2012), 99-107.